

Tips and tricks surviving the DMC-jungle¹,

or,

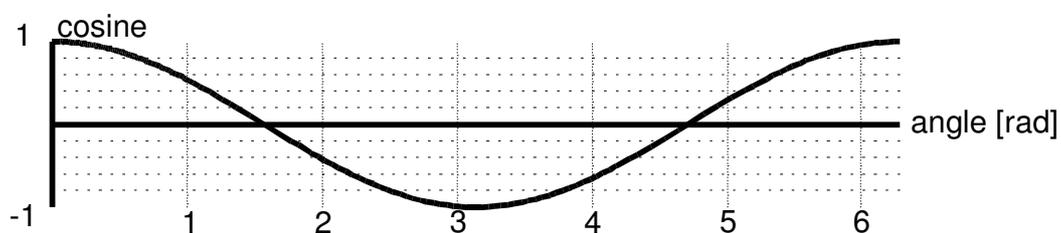
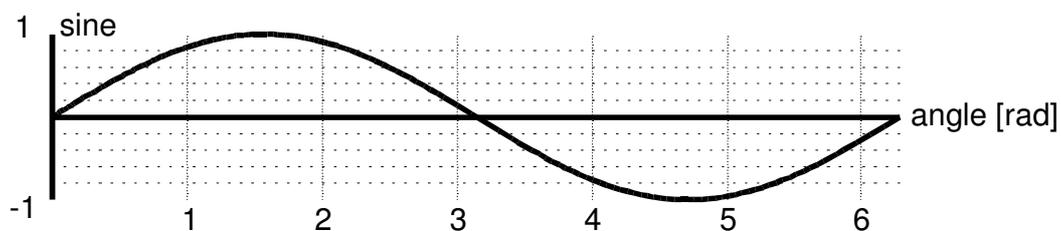
How do I imagine imaginary parts?

The course on Digital Motion Control is suited for everyone with some experience in the field of mechatronic systems. But this also means that people with very diverse professional background and education participate in this course. To smooth the differences between the participants we have prepared the following leaflet, combining answers to the most common questions on trigonometric functions, complex numbers and so on.

In our experience, most of this knowledge has been presented to the participants during courses on mathematics, dynamics and control theory, but has become somewhat dusted during the past years. Therefore we think it is sufficient to present the next subjects in a short abstract-like manner, as the “aha-effect” will surely revive many more fond memories.

Sine and Cosine

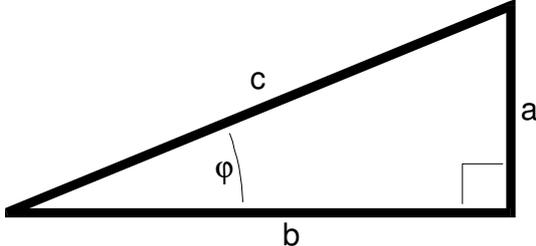
Sine and Cosine are two functions that define an angle. If this won't help you that much at this moment, the following picture will probably clarify a bit more.



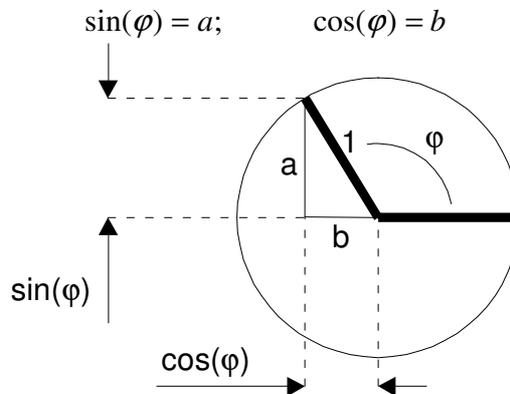
¹ This document is based on the same document provided by Philips CFT, which is gratefully acknowledged

From the picture it can be seen that both the Sine and the Cosine functions vary between values of -1 and 1, depending on the value of the angle. The angle is expressed in **radian** ([rad]), and an angle of 360 degrees (a full circle) equals a value of $2 \cdot \pi = 6.2832$ [rad]. One important aspect is the similarity between both functions. You can think about the cosine as a sine that starts a little bit later (more precisely at about an angle of $\pi/2 = 90$ degrees = 1.5708 rad.). This similarity is closely linked to the concept of **phase angles**, which will be discussed in detail in the course.

Better than just memorising graphs of both functions, you should understand how they can be constructed, using the **unitary circle**. The definitions for Sine and Cosine in a rectangular triangle are

$$\sin(\varphi) = \frac{a}{c}; \quad \cos(\varphi) = \frac{b}{c}$$


In the unitary circle, which has a radius of exactly one, a triangle can be constructed with the length of “c” equal to the radius. The definitions of Sine and Cosine then become:



Exercise: determine the approximate value of φ from the above two figures, and find the values of $\sin(\varphi)$ and $\cos(\varphi)$ that belong to this value from the graphs of Sine and Cosine.

Some important rules for calculating with Sine and Cosine apply to differentiating:

	Sine	Cosine
$f(\varphi)$	$\sin(\varphi)$	$\cos(\varphi)$
$\frac{\partial f(\varphi)}{\partial \varphi}$	$\cos(\varphi)$	$-\sin(\varphi)$

$\frac{\partial^2 f(\varphi)}{\partial \varphi^2}$	$-\sin(\varphi)$	$-\cos(\varphi)$
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Differentiating becomes very important when using Sine and Cosine functions for describing dynamic behaviour. As we will see further below, the free movement of a simple mass m on a spring c can be formulated as:

$$x(t) = A \sin(\omega_0 t + \phi)$$

with

$$\omega_0 = \sqrt{\frac{c}{m}}$$

By differentiating $x(t)$ with respect to time we get, using the above mathematical rules:

$$x(t) = A \sin(\omega_0 t + \phi)$$

$$v(t) = \omega_0 A \cos(\omega_0 t + \phi)$$

$$a(t) = -\omega_0^2 A \sin(\omega_0 t + \phi)$$

A more convenient way for calculating dynamic behaviour will be described in the next section.

Complex numbers

One of the more important mathematical concepts in Systems engineering is complex numbers. This is nothing more than an extension of the world of real numbers we all daily use. The only difference is the use of the **imaginary number “j”** (or “i”), which is defined as:

$$j = \sqrt{-1}$$

or

$$j^2 = -1$$

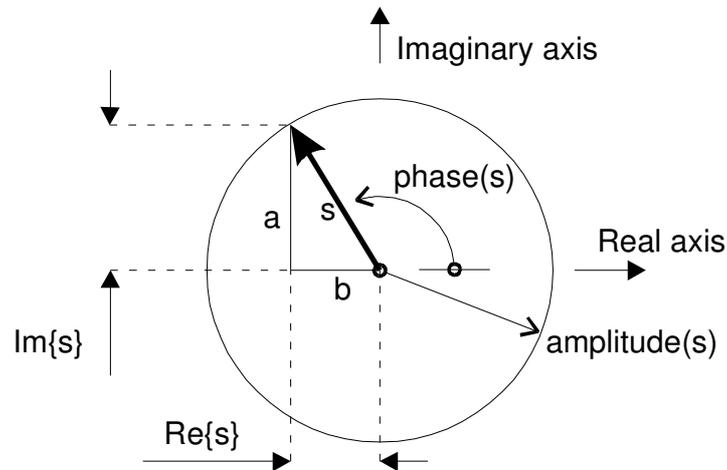
Generally a complex number will consist of a **real** and a **imaginary part**:

$$s = j^* a + b$$

$$\operatorname{Re}\{s\} = b$$

$$\operatorname{Im}\{s\} = a$$

Another way of looking at a complex number is considering its vector in the **complex plane**, and calculating the **amplitude** (length) and **angle** or **phase** with respect to the positive real axis of this vector. Look below for a picture of these important concepts:



$$s = j * a + b$$

$$\text{amplitude}(s) = |s| = \sqrt{a^2 + b^2}$$

$$\text{phase}(s) = \varphi(s) = \arctan\left(\frac{a}{b}\right)$$

The beauty of working with imaginary numbers is the simplicity of describing dynamic phenomena. By comparison with the unitary circle and using $\sin^2 + \cos^2 = 1$, we find the so-called **Euler notation**:

$$e^{j(\omega t + \phi)} = \sin(\omega t + \phi) + j * \cos(\omega t + \phi)$$

Using this, we can say that the motion of a mass on a spring (see above) can also be described as:

$$x(t) = A \sin(\omega t + \phi) = \text{Re}\{A e^{j(\omega t + \phi)}\}.$$

For calculating the velocity and acceleration of this system we can use the general rules for differentiating the exponential function:

$$f(x) = e^{jx}$$

$$\frac{\partial f(x)}{\partial t} = \frac{\partial e^{jx}}{\partial t} = e^{jx} \frac{\partial x}{\partial t}$$

It gets even better if we start looking at systems with damping included. If we consider only linear viscous damping (the magnitude of the damping force is only depending on the magnitude of the velocity), the free movement of a mass m on a spring c with a small damper d can be described as:

$$x(t) = e^{-\beta t} A \sin(\omega t + \phi)$$

with

$$\omega = \frac{\sqrt{4mc - d^2}}{2m}$$

$$\beta = \frac{d}{2m}$$

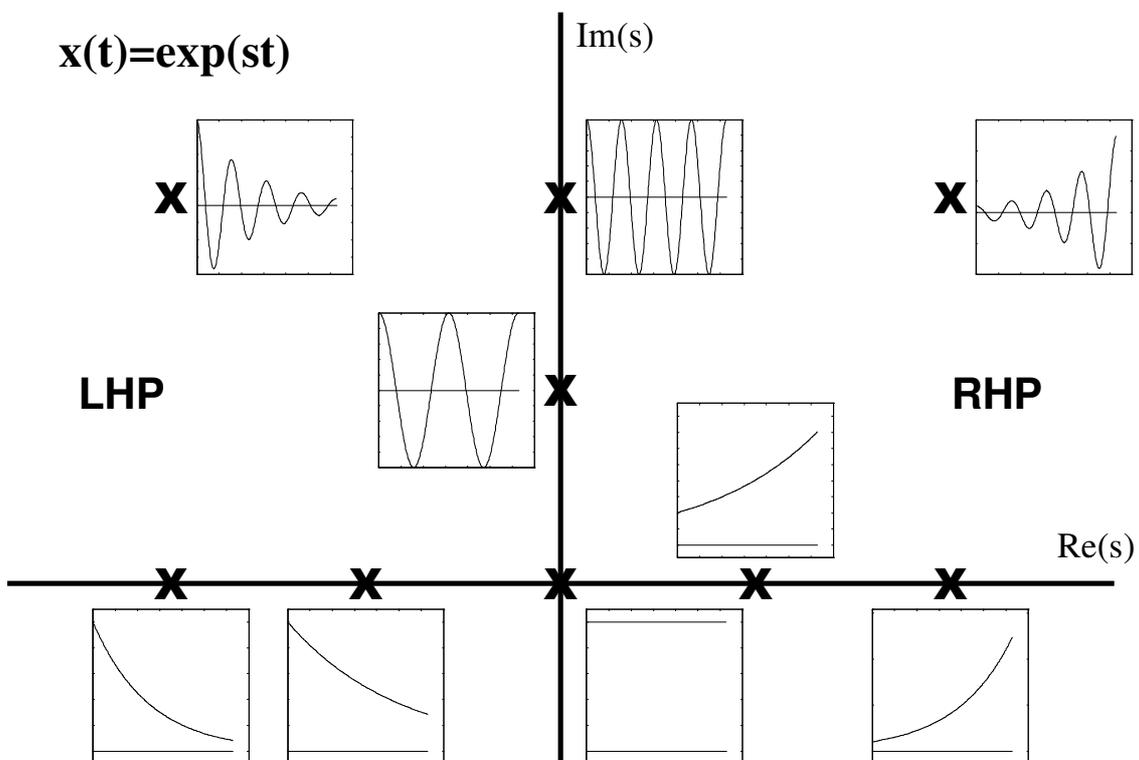
which can be transformed with the Euler notation to:

$$x(t) = e^{-\beta t} A \sin(\omega t + \phi) = e^{-\beta t} \operatorname{Re}\{Ae^{j(\omega t + \phi)}\} = \operatorname{Re}\{Ae^{-\beta t} e^{j(\omega t + \phi)}\} = \operatorname{Re}\{Ae^{j(\omega t + \phi) - \beta t}\}.$$

If we look at this expression in a more general manner, and forget for a moment the phase angle ϕ :

$$x(t) = \operatorname{Re}\{Ae^{(ja-b)t}\} = \operatorname{Re}\{Ae^{st}\}$$

From this we learn that the **real** part of s has something to do with the **damping** of a signal (large negative real part means large damping, large positive real part means negative damping, i.e. growing signal) and the **imaginary** part of s has something to do with the **frequency** of a oscillation. This can also be summarised in the following picture of the **complex plane**, with examples of several values of s and the belonging signals.



It is important to understand the meaning of s as a **parameter to describe the behaviour** of dynamic signals. A lot of the problems in understanding complex numbers in systems dynamics arise from the attempt to give s other meanings.

Just remember that s describes the frequency and damping of a signal, and memorise the above picture and you will survive the DMC-course (and much of your professional life thereafter) probably unharmed.

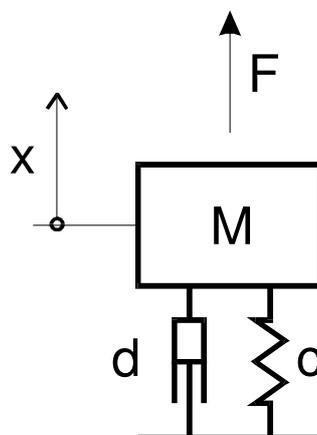
One last thing about the Left Half Plane (LHP), which has some magic around it. Looking at the above picture shows that signals with s -es that have a negative real part (which means s is in the left half of the complex plane) show a decreasing signal over time, which is normally desirable in technical applications, whereas signals with s -es in the right half plane show increasing signals over time, much like an explosion, which is normally not OK in devices that we want to use!

Mechanical systems

Most of the systems we have to control, and all of the systems in the DMC-course are electromechanical systems. In a large number of these systems the electrical and electronic part of the system is much faster than the mechanical part. Exceptions can be found in the field of precision engineering (e.g. IC-equipment) and fast moving parts (e.g. CD-optical pick-up units). But in most cases the mechanical part of a system determines the dynamic behaviour to good approximation.

Therefore it is important that a mathematical description of the dynamic behaviour of a mechanical system can be found, to help in designing controllers off-line (by simulation) and predicting machine-performance in advance.

We will not give you a full treatise on multi-body dynamic analysis, but just present you the steps to derive the equation of motion of a simple mechanical system with one degree of freedom (DOF).



To use Newton's Law (actio equals reactio) we look for all forces that act on the mass m . These are, besides the obvious external force F , the forces by spring c and damper d . The (vectorial) sum of all forces equals the change of impulse of the body, or easier said, the mass times the acceleration.

$$F_c = -c x; \quad \text{spring force,} \quad c = [N / m]$$

$$F_d = -d \dot{x}; \quad \text{damper force,} \quad d = [Ns / m]$$

$$m\ddot{x} = F - d \dot{x} - c x \rightarrow m\ddot{x} + d \dot{x} + c x = F.$$

During the course we will learn you how to use this knowledge to calculate a **frequency response function** (FRF) of a mechanical system, the basis of many control tuning strategies.

Db's

Our last subject, is also basically the most simple one. It will be not more than a definition, as every effort in making you understand the how's and why's of this definition will probably confuse more than give any clarification.

$$H[dB] = 20 \log\left(\frac{M_1}{M_2}\right)$$

M_1 and M_2 can have any meaning, like voltages, currents or movements. It is important to remember that dB's give a measure for a ratio of two values. With a dynamic signal analyser measuring the behaviour of a dynamic system, this will mostly be the ratio between the output of the system and the input of that system, but in principle every ratio is possible.

Some convenient values of H to remember are summarised in the following table:

H [dB]	Ratio
6	$M_1 = 2 M_2$
9.5	$M_1 = 3 M_2$
12	$M_1 = 4 M_2$
20	$M_1 = 10 M_2$
40	$M_1 = 100 M_2$